A RECURSIVE APPROACH TO THE EQUATIONS OF MOTION FOR THE MANEUVERING AND CONTROL OF FLEXIBLE MULTI-BODY SYSTEMS

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OVERVIEW

- The interest lies in a mathematical formulation capable of accommodating the problem of maneuvering a space structure consisting of a chain of articulated flexible substructures.

- Simultaneously, any perturbations from the "rigid-body" maneuvering and any elastic vibration must be suppressed.

- The equations of motion for flexible bodies undergoing rigid-body motions and elastic vibrations can be obtained conveniently by means of Lagrange's equations in terms of quasi-coordinates.

- The advantage of this approach is that it yields equations in terms of body axes, which are the same axes that are used to express the control forces and torques.

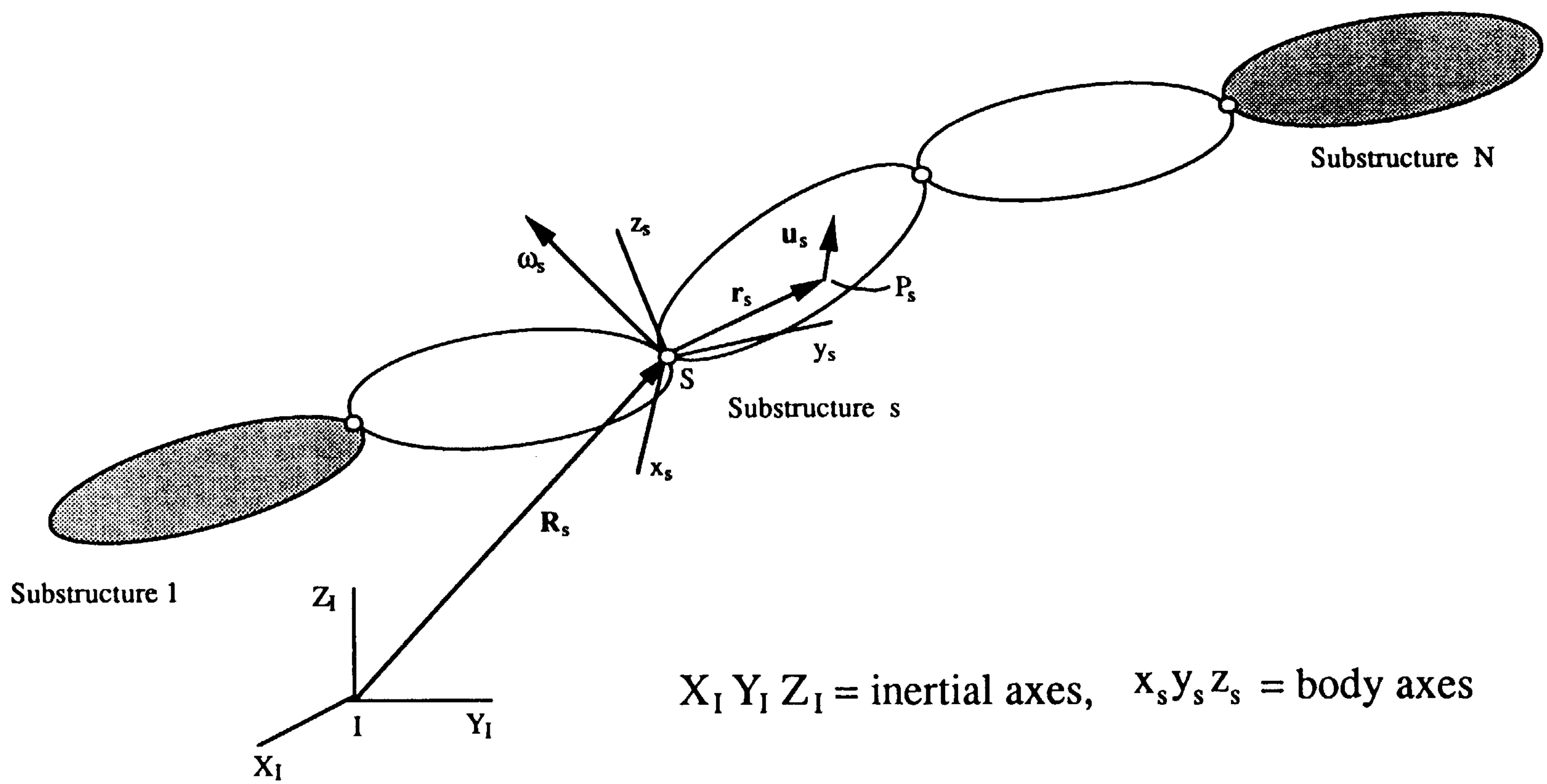
OVERVIEW (CONT'D)

- The equations of motion are nonlinear hybrid (ordinary and partial) differential equations.
- The partial differential equations can be discretized (in space) by means of the finite element method or the classical Rayleigh-Ritz method.
- The result is a set of nonlinear ordinary differential equations of high order.
- The nonlinearity can be traced to the rigid-body motions and the high order to the elastic vibration.
- Elastic motions tend to be small when compared with rigid-body motions.
- A perturbation approach permits breaking the problem into one for the rigid-body motions, which is nonlinear and of relatively low order, and for the elastic motions and the perturbations caused by them, which is linear and of relatively high order.

OVERVIEW (CONT'D)

- The rigid-body problem, which is associated with the maneuvering, is referred to as the zero-order (in a perturbation sense) problem and the control tends to be open loop.
- The perturbation suppression, which is associated with control, is referred to as the first-order problem and the control is closed-loop.
- The equations of motion are first derived for each individual substructure and then assembled into a single set for the fully interacting structure.
- The above is carried out by means of a kinematical synthesis eliminating the surplus coordinates.
- The kinematical synthesis, based on recursive relations, is carried out both for the zero-order and first-order problems.

HYBRID EQUATIONS FOR THE SUBSTRUCTURES



$X_I\,Y_I\,Z_I$ = inertial axes, $x_s y_s z_s$ = body axes

Figure 1 - The Articulated Chain of Substructures

HYBRID EQUATIONS FOR THE SUBSTRUCTURES

Derivation of the equations of motion by the Lagrangian approach requires the Lagrangian, and hence the kinetic and potential energy.

Position Vector of Point P_S in Substructure s :

$$\mathbf{W}_s(t) = \mathbf{R}_s(t) + \mathbf{r}_s(P_s) + \mathbf{u}_s(P_s, t), \quad s = 1, 2, ..., N$$

$\mathbf{R}_S$ = radius vector from I to S; in terms of inertial coordinates
$\mathbf{r}_S$ = radius vector from S to P_S; in terms of body coordinates

$\mathbf{u}_S$ = elastic displacement vector of P_S ; in terms of body coordinates

Velocity Vector of P_S:

$$\begin{aligned} \dot{\mathbf{W}}_s(t) &= \mathbf{V}_s(t) + \tilde{\omega}_s(t)(\mathbf{r}_s(P_s) + \mathbf{u}_s(P_s, t)) + \mathbf{v}_s(P_s, t) \\ &= \mathbf{V}_s(t) + (\tilde{r}_s(P_s) + \tilde{u}_s(P_s, t))^T \boldsymbol{\omega}_s(t) + \mathbf{v}_s(P_s, t), \quad s = 1, 2, \ldots, N \end{aligned}$$

$\mathbf{V}_S$ = velocity vector of S ; in terms of body coordinates
ω_S = absolute angular velocity vector of $x_S y_S z_S$; in terms of body coordinates
$\tilde{\omega}_S$ = skew symmetric matrix formed from ω_S
$\mathbf{v}_S = \dot{\mathbf{u}}_S$ = elastic velocity vector of P_S; in terms of body coordinates

HYBRID EQUATIONS FOR THE SUBSTRUCTURES(CONT'D)

Relation Between Inertial and Body-Axes Velocity Vectors:

$$\mathbf{V}_s = C_s \dot{\mathbf{R}}_s \ , \quad \boldsymbol{\omega}_s = D_s \dot{\boldsymbol{\theta}}_s$$

$C_s = C_s(\theta_{s1}, \theta_{s2}, \theta_{s3})$ = matrix of direction cosines between $x_s y_s z_s$ and $X_I Y_I Z_I$

$D_s = D_s(\theta_{s1}, \theta_{s2}, \theta_{s3})$ = transformation matrix

$\theta_{s1}, \theta_{s2}, \theta_{s3}$ = angles defining the orientation of $x_s y_s z_s$ and referred to $X_I Y_I Z_I$

$\mathbf{V}_s$, ω_s can be regarded as time derivatives of quasi-coordinates

HYBRID EQUATIONS FOR THE SUBSTRUCTURES(CONT'D)

Kinetic Energy:

$$\begin{aligned} T_s &= \frac{1}{2}\int_{D_s} \rho_s \dot{\mathbf{W}}_s^T \dot{\mathbf{W}}_s dD_s \\ &= \frac{1}{2} m_s \mathbf{V}_s^T \mathbf{V}_s + \mathbf{V}_s^T \tilde{S}_s^T \boldsymbol{\omega}_s + \mathbf{V}_s^T \int_{D_s} \rho_s \mathbf{v}_s dD_s \\ &\quad + \frac{1}{2}\boldsymbol{\omega}_s^T J_s \boldsymbol{\omega}_s + \boldsymbol{\omega}_s^T \int_{D_s} \rho_s (\tilde{r}_s + \tilde{u}_s)\mathbf{v}_s dD_s + \frac{1}{2}\int_{D_s} \rho_s \mathbf{v}_s^T \mathbf{v}_s dD_s \end{aligned}$$

$$m_s = \int_{D_s} \rho_s dD_s, \ \tilde{S}_s = \int_{D_s} \rho_s (\tilde{r}_s + \tilde{u}_s) dD_s, \ J_s = \int_{D_s} \rho_s (\tilde{r}_s + \tilde{u}_s)(\tilde{r}_s + \tilde{u}_s)^T dD_s$$

Potential Energy:

$$V_s = \frac{1}{2}[\mathbf{u}_s, \mathbf{u}_s]$$

ρ_S = mass density; D_S = domain of substructure s

[,] = energy inner product

HYBRID EQUATIONS FOR THE SUBSTRUCTURES(CONT'D)

General Hybrid Lagrange's Equations in Terms of Quasi-Coordinates:

$$\frac{d}{dt}\left(\frac{\partial L_s}{\partial \mathbf{V}_s}\right) + \tilde{\omega}_s \left(\frac{\partial L_s}{\partial \mathbf{V}_s}\right) - C_s \left(\frac{\partial L_s}{\partial \mathbf{R}_s}\right) = \mathbf{F}_s$$

$$\frac{d}{dt}\left(\frac{\partial L_s}{\partial \boldsymbol{\omega}_s}\right) + \tilde{V}_s \left(\frac{\partial L_s}{\partial \mathbf{V}_s}\right) + \tilde{\omega}_s \left(\frac{\partial L_s}{\partial \boldsymbol{\omega}_s}\right) - \left(D_s^T\right)^{-1} \left(\frac{\partial L_s}{\partial \boldsymbol{\theta}_s}\right) = \mathbf{M}_s$$

$$\frac{\partial}{\partial t}\left(\frac{\partial \hat{L}_s}{\partial \mathbf{v}_s}\right) - \left(\frac{\partial \hat{T}_s}{\partial \mathbf{u}_s}\right) + \mathcal{L}_s \mathbf{u}_s = \hat{\mathbf{U}}_s$$

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$L_S = T_S - V_S$ = Lagrangian; $\widehat{L}_S$ = Lagrangian density

$\widehat{T}_S$ = kinetic energy density; $\mathcal{L}$ = (stiffness) differential operator matrix

$\mathbf{F}_S$, $\mathbf{M}_S$ = resultant force and torque vectors

$\widehat{\mathbf{U}}_S$ = force density vector

HYBRID EQUATIONS FOR THE SUBSTRUCTURES(CONT'D)

Explicit Hybrid Equations:

$$m_s\dot{\mathbf{V}}_s + \tilde{S}_s^T\dot{\boldsymbol{\omega}}_s + \int_{D_s}\rho_s\dot{\mathbf{v}}_s dD_s = (2\tilde{S}_{vs} + m_s\tilde{V}_s + \tilde{\omega}_s\tilde{S}_s)\boldsymbol{\omega}_s + C_s\left(\frac{\partial L_s}{\partial \mathbf{R}_s}\right) + \mathbf{F}_s$$

$$\tilde{S}_s\dot{\mathbf{V}}_s + J_s\dot{\boldsymbol{\omega}}_s + \int_{D_s}\rho_s(\tilde{r}_s+\tilde{u}_s)\dot{\mathbf{v}}_s dD_s = (\tilde{S}_s\tilde{V}_s - \tilde{\omega}_s J_s - J_{vs})\boldsymbol{\omega}_s - \tilde{\omega}_s\int_{D_s}\rho_s(\tilde{r}_s+\tilde{u}_s)\mathbf{v}_s dD_s + (D_s^T)^{-1}\left(\frac{\partial L_s}{\partial \boldsymbol{\theta}_s}\right) + \mathbf{M}_s$$

$$\rho_s[\dot{\mathbf{V}}_s + (\tilde{r}_s + \tilde{u}_s)^T\dot{\boldsymbol{\omega}}_s + \dot{\mathbf{v}}_s] = \rho_s(\tilde{V}_s + 2\tilde{v})\boldsymbol{\omega}_s - \rho_s\tilde{\omega}_s^2(\mathbf{r}_s + \mathbf{u}_s) - \mathcal{L}_s\mathbf{u}_s + \hat{\mathbf{U}}_s$$

where

$$S_{vs} = \dot{S}_s = \int_{D_s}\rho_s\tilde{v}_s dD_s\ ,\quad J_{vs} = \dot{J}_s = \int_{D_s}\rho_s[\tilde{v}_s(\tilde{r}_s + \tilde{u}_s)^T + (\tilde{r}_s + \tilde{u}_s)\tilde{v}_s^T]dD_s$$

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Augmenting Equations:

$$\dot{\mathbf{R}}_s = C_s^T\mathbf{V}_s\ ,\quad \dot{\boldsymbol{\theta}}_s = D_s^{-1}\boldsymbol{\omega}_s\ ,\quad \dot{\mathbf{u}}_s = \mathbf{v}_s$$

ORDINARY DIFFERENTIAL EQUATIONS

Elastic Displacement Vector: $\mathbf{u}_s(P_s,t) = \Phi_s(P_s)\mathbf{q}_s(t), s = 1,2,\ldots,N$

Φ_S = matrix of admissible functions (shape functions)

$\mathbf{q}_S$ = vector of generalized displacements

- Derive discretized T_S and V_S

Discretized State Equations:

$$m_s\dot{\mathbf{V}}_s + \tilde{S}_{s0}^T\dot{\boldsymbol{\omega}}_s + \bar{\Phi}_s\dot{\mathbf{p}}_s = -m_s\tilde{\omega}_s\mathbf{V}_s - \tilde{\omega}_s\tilde{S}_{s0}^T\boldsymbol{\omega}_s - 2\tilde{\omega}_s\bar{\Phi}_s\mathbf{p}_s - (\dot{\tilde{\omega}}_s + \tilde{\omega}_s^2)\bar{\Phi}_s\mathbf{q}_s + \mathbf{F}_s$$

$$\tilde{S}_{s0}\dot{\mathbf{V}}_s + J_{s0}\dot{\boldsymbol{\omega}}_s + \tilde{\Phi}_s\dot{\mathbf{p}}_s = -\tilde{S}_{s0}\tilde{\omega}_s\mathbf{V}_s - \tilde{\omega}_s J_{s0}\boldsymbol{\omega}_s - 2\hat{\Phi}_s\mathbf{p}_s - [(\widetilde{[\tilde{V}_s\boldsymbol{\omega}_s]} - \dot{\tilde{V}}_s)\bar{\Phi}_s + 2\dot{\hat{\Phi}}_s + 2\tilde{\omega}_s\hat{\Phi}_s - (\dot{\tilde{\omega}}_s + \tilde{\omega}_s^2)\tilde{\Phi}_s]\mathbf{q}_s + \mathbf{M}_s$$

$$\bar{\Phi}_s^T\dot{\mathbf{V}}_s + \tilde{\Phi}_s^T\dot{\omega}_s + M_s\dot{\mathbf{p}}_s = -\bar{\Phi}_s^T\tilde{\omega}_s\mathbf{V}_s + \hat{\Phi}_s^T\omega_s - 2\tilde{H}_s\mathbf{p}_s - [K_s + \bar{H}_s(\omega_s) + \dot{\tilde{H}}_s)]\mathbf{q}_s + \mathbf{Q}_s$$

Various terms involve integrals over D_S

Augmenting Equations: $\dot{\mathbf{R}}_s = C_s^T\mathbf{V}_s\ ,\ \dot{\boldsymbol{\theta}}_s = D_s^{-1}\boldsymbol{\omega}_s\ ,\ \dot{\mathbf{q}}_s = \mathbf{p}_s$

PERTURBATION APPROACH

Perturbation Expansions:

$$\mathbf{V}_s = \mathbf{V}_{s0} + \mathbf{V}_{s1} \ , \ \boldsymbol{\omega}_s = \boldsymbol{\omega}_{s0} + \boldsymbol{\omega}_{s1} \ , \ \mathbf{F}_s = \mathbf{F}_{s0} + \mathbf{F}_{s1} \ , \ \mathbf{M}_s = \mathbf{M}_{s0} + \mathbf{M}_{s1}$$

- Subscript 0 denotes zero-order (in a perturbation sense) quantities
- Subscript 1 denotes first-order quantities
- First-order terms are one order of magnitude smaller than zero-order terms
- Elastic displacements and velocities are by definition of first order.

PERTURBATION APPROACH(CONT'D)

Introduce perturbation expansion into state equations and separate orders of magnitude.

Zero-Order State Equations: $\mathcal{M}_{s0}\dot{\mathbf{x}}_{s0} = \mathcal{C}_{s0}\mathbf{x}_{s0} + \mathcal{B}_{s0}\mathbf{f}_{s0} + \mathcal{D}_{s0}\mathbf{d}_{s0}$

Zero-Order State and Excitation Vectors:

$$\mathbf{x}_{s0}(t) = [\mathbf{R}_{s0}^T(t)\ \boldsymbol{\theta}_{s0}^T(t)\ \mathbf{V}_{s0}^T(t)\ \boldsymbol{\omega}_{s0}^T(t)]^T\ ,\ \ \mathbf{f}_{s0}(t) = [\mathbf{F}_{s0}^T(t)\ \mathbf{M}_{s0}^T(t)]^T$$

Coefficient Matrices:

$$\mathcal{M}_{s0} = \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & m_s I & \tilde{S}_{s0}^T \\ 0 & 0 & \tilde{S}_{s0} & J_{s0} \end{bmatrix},\ \ \mathcal{C}_{s0} = \begin{bmatrix} 0 & 0 & C_{s0}^T & 0 \\ 0 & 0 & 0 & D_{s0}^{-1} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\mathcal{B}_{s0} = \mathcal{D}_{s0} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ I & 0 \\ 0 & I \end{bmatrix},\ \ \mathbf{d}_{s0} = \begin{bmatrix} -m_s\tilde{\omega}_{s0}\mathbf{V}_{s0} - \tilde{\omega}_{s0}\tilde{S}_{s0}^T\boldsymbol{\omega}_{s0} \\ -\tilde{S}_{s0}\tilde{\omega}_{s0}\mathbf{V}_{s0} - \tilde{\omega}_{s0}J_{s0}\boldsymbol{\omega}_{s0} \end{bmatrix}$$

PERTURBATION APPROACH(CONT'D)

First-Order State Equations: $\mathcal{M}_{s1}\dot{\mathbf{x}}_{s1} = \mathcal{C}_{s1}\mathbf{x}_{s1} + \mathcal{B}_{s1}\mathbf{f}_{s1} + \mathcal{D}_{s1}\mathbf{d}_{s1}$

First-Order State and Excitation Vectors:

$$\mathbf{x}_{s1}(t) = [\mathbf{U}_{s1}^T(t)\ \beta_{s1}^T(t)\ \mathbf{q}_s^T(t)\ \mathbf{V}_{s1}^T(t)\ \omega_{s1}^T(t)\ \mathbf{p}_s^T(t)]^T\ ,\quad \mathbf{f}_{s1}(t) = [\mathbf{F}_{s1}^T(t)\ \mathbf{M}_{s1}^T(t)\ \mathbf{Q}_s^T(t)]^T$$

$\mathbf{U}_{s1}$ = body-axes vector of perturbations in translational displacements

β_{s1} = body-axes vectors of perturbations in rotational displacements

Coefficient Matrices:

$$\mathcal{M}_{s1} = \begin{bmatrix} I & 0 & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 & 0 \\ 0 & 0 & 0 & m_s I & \tilde{S}_{s0}^T & \bar{\Phi}_s \\ 0 & 0 & 0 & \tilde{S}_{s0} & J_{s0} & \tilde{\Phi}_s \\ 0 & 0 & 0 & \bar{\Phi}_s^T & \tilde{\Phi}_s^T & M_s \end{bmatrix}$$

PERTURBATION APPROACH(CONT'D)

Coefficient Matrices:

$$\mathcal{C}_{s1} = \begin{bmatrix} -\tilde{\omega}_{s0} & -\tilde{V}_{s0} & 0 & I & 0 & 0 \\ 0 & -\tilde{\omega}_{s0} & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & 0 & I \\ 0 & 0 & -(\tilde{\dot{\omega}}_{s0} + \tilde{\omega}_{s0}^2)\bar{\Phi}_{s0} & -m_s\tilde{\omega}_{s0} & -\Gamma_s & -2\tilde{\omega}_{s0}\bar{\Phi}_s \\ 0 & 0 & -\Xi_s & -\tilde{S}_s\tilde{\omega}_{s0} & -\Delta_s & -2\hat{\Phi}_s \\ 0 & 0 & -\bar{K}_s & -\bar{\Phi}_s^T\tilde{\omega}_{s0} & -\Upsilon_s & -2\tilde{H}_s \end{bmatrix}$$

$$\mathcal{B}_{s1} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}, \quad \mathcal{D}_{s1} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ I \end{bmatrix}$$

in which

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$$\Gamma_s = \tilde{S}_{s0}\tilde{\omega}_{s0} - 2\tilde{\omega}_{s0}\tilde{S}_{s0} - m_s\tilde{V}_{s0}\ , \quad \Delta_s = 2\tilde{\omega}_{s0}J_{s0} + J_{s0}\tilde{\omega}_{s0} - (trJ_{s0})\tilde{\omega}_{s0} - \tilde{S}_{s0}\tilde{V}_{s0}$$

$$\Xi_s = ([\widetilde{\tilde{V}_{s0}\omega_{s0}}] - \tilde{\dot{V}}_{s0})\bar{\Phi}_s + 2\dot{\hat{\Phi}}_s + 2\tilde{\omega}_s\hat{\Phi}_s - (\tilde{\dot{\omega}}_{s0} + \tilde{\omega}_{s0}^2)\tilde{\Phi}_s$$

$$\Upsilon_s = \tilde{\Phi}_s^T\tilde{\omega}_{s0}^T - 2\hat{\Phi}_s^T + \bar{\Phi}_s^T\tilde{V}_{s0}^T,\ \bar{K}_s = K_s + \bar{H}_s + \dot{\tilde{H}}_s$$

$$\mathbf{d}_{s1} = -\bar{\Phi}_s^T(\dot{\mathbf{V}}_{s0} + \tilde{\omega}_{s0}\mathbf{V}_{s0}) - \tilde{\Phi}_s\dot{\boldsymbol{\omega}}_{s0} - \hat{\Phi}_s^T\boldsymbol{\omega}_{s0}$$

KINEMATICAL SYNTHESIS FOR ZERO-ORDER EQUATIONS

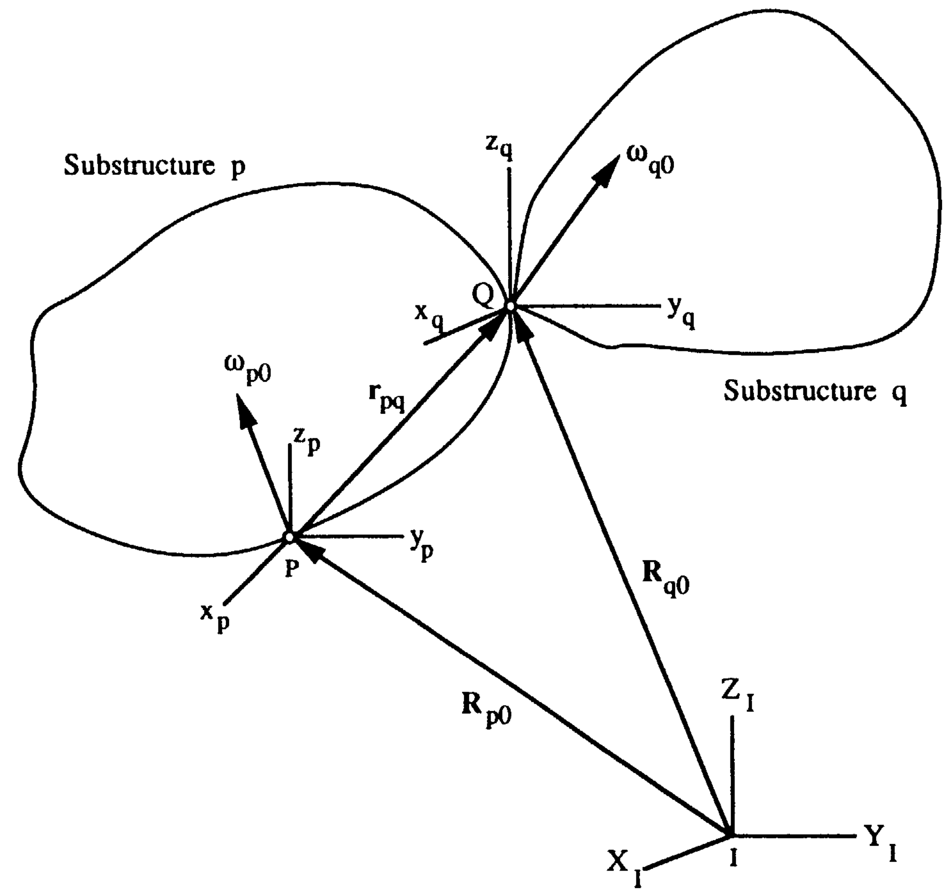

Figure 2 - Two Adjacent Substructures in the Chain

KINEMATICAL SYNTHESIS FOR ZERO-ORDER EQUATIONS (CONT'D)

Kinematical Constraints (linking the substructure together):

$$\mathbf{R}_{q0} = \mathbf{R}_{p0} + C_{p0}^T \mathbf{r}_{pq} \ , \quad \boldsymbol{\theta}_{q0} = \boldsymbol{\theta}_{q0}$$

$$\mathbf{V}_{q0} = C_{qp}\mathbf{V}_{p0} - C_{qp}\tilde{r}_{pq}\boldsymbol{\omega}_{p0} \ , \quad \boldsymbol{\omega}_{q0} = \boldsymbol{\omega}_{q0}$$

Recursive Relations:

$$\mathbf{R}_{s0} = \mathbf{R}_{10} + \sum_{j=2}^{s} C_{j-1,0}^T \mathbf{r}_{j-1,j}$$

$$\mathbf{V}_{s0} = \prod_{j=s}^{2} C_{j,j-1}\mathbf{V}_{10} - \prod_{j=s}^{2} C_{j,j-1}\tilde{r}_{12}\omega_{10} - \prod_{j=s}^{3} C_{j,j-1}\tilde{r}_{23}\omega_{20} \cdots - C_{s,s-1}\tilde{r}_{s-1,s}\omega_{s0}$$

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$$\mathbf{x}_{s0}^u = T_{s0}\mathbf{x}_{s0}^c + \bar{C}_s \mathbf{r}_{s0} \ , \quad s = 1, 2, \cdots, N$$

$$\mathbf{x}_{s0}^u = \mathbf{x}_{s0} \, , \ \mathbf{x}_{s0}^c = [\mathbf{R}_{10}^T \, \boldsymbol{\theta}_{10}^T \, \boldsymbol{\theta}_{20}^T \ \cdots \ \boldsymbol{\theta}_{s0}^T \, \mathbf{V}_{10}^T \, \boldsymbol{\omega}_{10}^T \, \boldsymbol{\omega}_{20}^T \ \cdots \ \boldsymbol{\omega}_{s0}^T]^T \ , \quad s = 1, 2, \cdots, N$$

$$\mathbf{r}_{s0} = [\mathbf{r}_{12}^T \, \mathbf{r}_{23}^T \ \cdots \ \mathbf{r}_{s-1,s}^T]^T \ , \quad s = 1, 2, \cdots, N$$

KINEMATICAL SYNTHESIS FOR ZERO-ORDER EQUATIONS (CONT'D)

$$\mathcal{M}_0 \dot{\mathbf{x}}_0^u = \mathcal{C}_0 \mathbf{x}_0^u + \mathcal{B}_0 \mathbf{f}_0 + \mathcal{D}_0 \mathbf{d}_0$$

Disjoint State Equations:

$$\mathbf{x}_0^u = \begin{bmatrix} \mathbf{x}_{10} \\ \mathbf{x}_{20} \\ \vdots \\ \mathbf{x}_{N0} \end{bmatrix}, \quad \mathbf{f}_0 = \begin{bmatrix} \mathbf{f}_{10} \\ \mathbf{f}_{20} \\ \vdots \\ \mathbf{f}_{N0} \end{bmatrix}, \quad \mathbf{d}_0 = \begin{bmatrix} \mathbf{d}_{10} \\ \mathbf{d}_{20} \\ \vdots \\ \mathbf{d}_{N0} \end{bmatrix}$$

Completion of the State Dimension:

$$\mathbf{x}_{10}^* = \begin{bmatrix} \mathbf{x}_{10} \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad \mathbf{x}_{20}^* = \begin{bmatrix} 0 \\ \mathbf{x}_{20} \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \ldots, \quad \mathbf{x}_{N0}^* = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ \mathbf{x}_{N0} \end{bmatrix}$$

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Unconstrained State: $\mathbf{x}_0^u = \sum_{s=1}^{N} \mathbf{x}_{s0}^*$

Full-Dimension Constraint Equation:

$$\mathbf{x}_0^u = \sum_{s=1}^{N} \mathbf{x}_{s0}^* = \sum_{s=1}^{N} T_{s0}^* \mathbf{x}_0^c + \bar{C}\mathbf{r}_0 = T_0 \mathbf{x}_0^c + \bar{C}\mathbf{r}_0$$

Constrained State:

$$\mathbf{x}_0^c = \mathbf{x}_{N0}^c = [\mathbf{R}_{10}^T \; \boldsymbol{\theta}_{10}^T \; \boldsymbol{\theta}_{20}^T \; \cdots \; \boldsymbol{\theta}_{N0}^T \; \mathbf{V}_{10}^T \; \boldsymbol{\omega}_{10}^T \; \boldsymbol{\omega}_{20}^T \; \cdots \; \boldsymbol{\omega}_{N0}^T]^T, \quad \mathbf{r}_0 = \mathbf{r}_{N0}$$

KINEMATICAL SYNTHESIS FOR ZERO-ORDER EQUATIONS (CONT'D)

State Equations for Zero-Order Problems:

$$\dot{\mathbf{x}}_0 = \mathcal{A}_0\mathbf{x}_0 + \mathcal{B}_0^*\mathbf{f}_0 + \mathcal{D}_0^*\mathbf{d}_0 + \mathcal{R}_0^*\mathbf{r}_0$$

$$\mathcal{A}_0 = \left(T_0^T\mathcal{M}_0T_0\right)^{-1}T_0^T\left(\mathcal{C}_0T_0 - \mathcal{M}_0\dot{T}_0\right), \quad \mathcal{B}_0^* = \left(T_0^T\mathcal{M}_0T_0\right)^{-1}T_0^T\mathcal{B}_0$$

$$\mathcal{D}_0^* = \left(T_0^T\mathcal{M}_0T_0\right)^{-1}T_0^T\mathcal{D}_0, \quad \mathcal{R}_0^* = \left(T_0^T\mathcal{M}_0T_0\right)^{-1}T_0^T\left(\mathcal{C}_0\bar{C} - \mathcal{M}_0\dot{\bar{C}}\right)$$

Note: Superscript c was dropped for simplicity

KINEMATICAL SYNTHESIS FOR FIRST-ORDER EQS.

Kinematical Constraints Yield Recursive Relations:

$$\mathbf{U}_{q1} = C_{qp}[\mathbf{U}_{p1} - \tilde{r}_{pq}\boldsymbol{\beta}_{p1} + \Phi_{pq}\mathbf{q}_p] \,, \quad \boldsymbol{\beta}_{q1} = C_{qp}(\boldsymbol{\beta}_{p1} + \Psi_{pq}\mathbf{q}_p) \,, \quad \mathbf{q}_q = \mathbf{q}_q$$

$$\mathbf{V}_{q1} = [\tilde{V}_{p0}C_{qp} + C_{qp}(\widetilde{[\tilde{r}_{pq}\omega_{p0}]} - \tilde{V}_{p0})]\boldsymbol{\beta}_{p1} + [\tilde{V}_{p0}C_{qp}\Psi_{pq} + C_{qp}\tilde{\omega}_{p0}\Phi_{pq}]\mathbf{q}_p + C_{qp}\mathbf{V}_{p1} - C_{qp}\tilde{r}_{pq}\boldsymbol{\omega}_{p1} + C_{qp}\Phi_{pq}\mathbf{p}_p$$

$$\boldsymbol{\omega}_{q1} = C_{qp}(\tilde{\omega}_{p0}\Psi_{pq}\mathbf{q}_p + \boldsymbol{\omega}_{p1} + \Psi_{pq}\mathbf{p}_p) \,, \quad \mathbf{p}_q = \mathbf{p}_q$$

where

$$\Phi_{pq} = \Phi_p(\mathbf{r}_{pq}) \,, \quad \Psi_{pq} = \nabla \times \Phi_p(\mathbf{r}_{pq})$$

Matrix Form of Recursive Relations:

$$\mathbf{x}^u_{s1} = T_{s1}\mathbf{x}^c_{s1} \,, \quad s = 2, 3, \cdots, N$$

Disjoint Perturbation State Equations:

$$\mathcal{M}_1\dot{\mathbf{x}}^u_1 = \mathcal{C}_1\mathbf{x}^u_1 + \mathcal{B}_1\mathbf{f}_1 + \mathcal{D}_1\mathbf{d}_1$$

$$\mathbf{x}^u_1 = \begin{bmatrix} \mathbf{x}_{11} \\ \mathbf{x}_{21} \\ \vdots \\ \mathbf{x}_{N1} \end{bmatrix} , \quad \mathbf{f}_1 = \begin{bmatrix} \mathbf{f}_{11} \\ \mathbf{f}_{21} \\ \vdots \\ \mathbf{f}_{N1} \end{bmatrix} , \quad \mathbf{d}_1 = \begin{bmatrix} \mathbf{d}_{11} \\ \mathbf{d}_{21} \\ \vdots \\ \mathbf{d}_{N1} \end{bmatrix}$$

KINEMATICAL SYNTHESIS FOR FIRST-ORDER EQS. (CONT'D)

Completion of the Perturbation State Dimension:

$$\mathbf{x}_{11}^* = \begin{bmatrix} \mathbf{x}_{11} \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad \mathbf{x}_{21}^* = \begin{bmatrix} 0 \\ \mathbf{x}_{21} \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad \mathbf{x}_{31}^* = \begin{bmatrix} 0 \\ 0 \\ \mathbf{x}_{31} \\ \vdots \\ 0 \end{bmatrix}, \quad \mathbf{x}_{N1}^* = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ \mathbf{x}_{N1} \end{bmatrix}$$

Constrained Perturbation State Vector:

$$\mathbf{x}_1^c = [\mathbf{U}_{11}^T \; \boldsymbol{\beta}_{11}^T \; \mathbf{q}_1^T \; \mathbf{q}_2^T \; \cdots \; \mathbf{q}_N^T \; \mathbf{V}_{11}^T \; \boldsymbol{\omega}_{11}^T \; \mathbf{p}_1^T \; \mathbf{p}_2^T \; \cdots \; \mathbf{p}_N^T]^T$$

Full-Dimensional Constraint Equation:

$$\mathbf{x}_1^u = \sum_{s=1}^{N} \mathbf{x}_{s1}^* = \sum_{s=1}^{N} \mathcal{T}_{s1}^* \mathbf{x}_1^c = \mathcal{T}_1 \mathbf{x}_1^c$$

State Equations for First-Order Problem:

$$\dot{\mathbf{x}}_1 = \mathcal{A}_1 \mathbf{x}_1 + \mathcal{B}_1^* \mathbf{f}_1 + \mathcal{D}_1^* \mathbf{d}_1$$

$$\mathcal{A}_1 = (\mathcal{T}_1^T \mathcal{M}_1 \mathcal{T}_1)^{-1} \mathcal{T}_1^T (\mathcal{C}_0 \mathcal{T}_0 - \mathcal{M}_0 \dot{\mathcal{T}}_0), \quad \mathcal{B}_1^* = (\mathcal{T}_1^T \mathcal{M}_1 \mathcal{T}_1)^{-1} \mathcal{T}_1^T \mathcal{B}_1$$

$$\mathcal{D}_1^* = (\mathcal{T}_1^T \mathcal{M}_1 \mathcal{T}_1)^{-1} \mathcal{T}_1^T \mathcal{D}_1$$

SUMMARY AND CONCLUSIONS

- The equations of motion for a structure in the form of a collection of articulated flexible substructures can be derived conveniently by means of Lagrange's equations in terms of quasi-coordinates for flexible bodies.

- For practical reasons, the set of nonlinear hybrid (ordinary and partial) differential equations is transformed into a set of nonlinear ode's of high dimension.

- Due to the nature of the problem, a perturbation approach can be used to divide the equations into two sets containing terms differing in magnitude.

- The zero-order problem is nonlinear and of relatively low order. It is associated with the "rigid-body" maneuvering and the control is open loop.

SUMMARY AND CONCLUSIONS (CONT'D)

- The first-order problem is linear and of relatively high order. It is associated with the elastic vibration and the perturbation it causes in the rigid-body motions and the control is closed loop.

- The equations of motion are derived for each substructure separately.

- A given kinematical synthesis is used to link the various substructures together.

- The constraint equations lead to recursive relations that are used to eliminate the surplus coordinates.

- The procedure is used to derive state equations both for the zero-order and first-order problems.

- The formulation is particularly well suited for control design.